

A simple proof of the Ohsawa-Takegoshi extension theorem

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One of the most beautiful result in complex analysis is the following Ohsawa-Takegoshi extension theorem (cf. [6], see also [1], [3], [5], [7]):

Theorem. *Let Ω be a bounded pseudoconvex domain in \mathbb{C}^n . Suppose $\sup_{\Omega} |z_n|^2 < e^{-1}$. Then there exists a constant $C_n > 0$ such that for every $\varphi \in PSH(\Omega)$, every holomorphic function f on $\Omega \cap \{z_n = 0\}$ with $\int_{\Omega \cap \{z_n=0\}} |f|^2 e^{-\varphi} < \infty$, there exists a holomorphic extension F of f to Ω such that*

$$\int_{\Omega} \frac{|F|^2}{|z_n|^2 (-\log |z_n|)^2} e^{-\varphi} \leq C_n \int_{\Omega \cap \{z_n=0\}} |f|^2 e^{-\varphi}.$$

Recently, there are some attempts to simplify the original proof of Ohsawa-Takegoshi, which is based on a solution of certain *twisted* $\bar{\partial}$ -equation (cf. [8], [9]). In this paper, we shall give a simple proof by solving directly the $\bar{\partial}$ -equation. The idea is inspired by a remarkable paper of Berndtsson-Charpentier (cf. [2]).

Let $\rho = \log(|z_n|^2 + \epsilon^2)$, $\eta = -\rho + \log(-\rho)$ and $\psi = -\log \eta$, where $\epsilon > 0$ is a sufficiently small constant such that $-\rho \geq 1$ on Ω . Since

$$\partial \bar{\partial} \psi = -\frac{\partial \bar{\partial} \eta}{\eta} + \frac{\partial \eta \bar{\partial} \eta}{\eta^2} = (1 + (-\rho)^{-1}) \frac{\partial \bar{\partial} \rho}{\eta} + \frac{\partial \rho \bar{\partial} \rho}{\eta \rho^2} + \frac{\partial \eta \bar{\partial} \eta}{\eta^2}, \quad (1)$$

we have $\psi \in PSH(\Omega)$. Put $\phi = \varphi + \log |z_n|^2$. Let $\chi : \mathbb{R} \rightarrow [0, 1]$ be a C^∞ cut-off function satisfying $\chi|_{(-\infty, 1/2)} = 1$ and $\chi|_{(1, \infty)} = 0$. By a standard approximation argument we may assume that f is holomorphic in some domain $V \supset \supset \Omega \cap \{z_n = 0\}$, φ is C^∞ in a neighborhood of $\bar{\Omega}$, and it suffices to find a holomorphic extension F of f to Ω such that $\int_{\Omega} \frac{|F|^2}{|z_n|^2 (-\log |z_n|)^2} e^{-\varphi} \leq C_n \int_V |f|^2 e^{-\varphi}$. Thus for ϵ small enough, we have a well-defined smooth $\bar{\partial}$ -closed $(0, 1)$ form $v = f \bar{\partial} \chi(|z_n|^2 / \epsilon^2)$ on Ω . Clearly, $v \in L^2_{(0,1)}(\Omega, \phi)$ and there exists a solution of $\bar{\partial} u = v$ with minimal L^2 -norm in $L^2(\Omega, \phi)$, i.e., $u \perp \text{Ker } \bar{\partial}$. Since ψ is a bounded function, we have $ue^\psi \perp \text{Ker } \bar{\partial}$ in $L^2(\Omega, \phi + \psi)$. Thus by Hörmander's L^2 -estimates for the $\bar{\partial}$ operator (cf. [4]),

$$\begin{aligned} \int_{\Omega} |u|^2 e^{\psi-\phi} &\leq \int_{\Omega} |\bar{\partial}(ue^\psi)|^2_{\bar{\partial}\bar{\partial}(\phi+\psi)} e^{-\psi-\phi} \\ &= \int_{\Omega} |v + \bar{\partial}\psi \wedge u|^2_{\bar{\partial}\bar{\partial}(\phi+\psi)} e^{\psi-\phi} \\ &\leq (1 + r^{-1}) \int_{\Omega} |v|^2_{\bar{\partial}\bar{\partial}(\phi+\psi)} e^{\psi-\phi} + \int_{\Omega} |\bar{\partial}\psi|^2_{\bar{\partial}\bar{\partial}(\phi+\psi)} |u|^2 e^{\psi-\phi} \\ &\quad + r \int_{\text{supp } v} |\bar{\partial}\psi|^2_{\bar{\partial}\bar{\partial}(\phi+\psi)} |u|^2 e^{\psi-\phi} \quad (\text{by Schwarz's inequality}) \end{aligned} \quad (2)$$

where $r > 0$ is a small constant to be determined later. Since $\partial\eta\bar{\partial}\eta = (1 + (-\rho)^{-1})^2\partial\rho\bar{\partial}\rho$, we infer from (1) that

$$\partial\bar{\partial}\psi \geq \frac{\partial\rho\bar{\partial}\rho}{\eta\rho^2} + \frac{\partial\eta\bar{\partial}\eta}{\eta^2} = \left(\frac{1}{\eta^2} + \frac{1}{\eta(-\rho+1)^2}\right)\partial\eta\bar{\partial}\eta.$$

Thus

$$\int_{\Omega} |\bar{\partial}\psi|_{\partial\bar{\partial}(\phi+\psi)}^2 |u|^2 e^{\psi-\phi} \leq \int_{\Omega} \frac{|u|^2}{1 + \frac{\eta}{(-\rho+1)^2}} e^{\psi-\phi}. \quad (3)$$

By (1), we have $\partial\bar{\partial}\psi \geq \frac{\partial\bar{\partial}\rho}{\eta} = \frac{\epsilon^2 dz_n d\bar{z}_n}{\eta(|z_n|^2 + \epsilon^2)^2}$. Thus by Fubini's theorem, if $\epsilon \ll 1$, we have

$$\begin{aligned} \int_{\Omega} |v|_{\partial\bar{\partial}(\phi+\psi)}^2 e^{\psi-\phi} &\leq 2 \left(\int_{\{\frac{\epsilon^2}{2} < |z_n|^2 < \epsilon^2\}} |\chi'|^2 \frac{(|z_n|^2 + \epsilon^2)^2 |z_n|^2}{\epsilon^2} \frac{1}{\epsilon^4 |z_n|^2} \right) \int_V |f|^2 e^{-\varphi} \\ &\leq C_n \int_V |f|^2 e^{-\varphi}. \end{aligned} \quad (4)$$

Since $\partial\psi\bar{\partial}\psi = \frac{1}{\eta^2} \left(1 + \frac{1}{-\rho}\right)^2 \partial\rho\bar{\partial}\rho \leq \frac{4}{\eta^2} \partial\rho\bar{\partial}\rho$ and $\partial\bar{\partial}\psi \geq \frac{\epsilon^2 dz_n d\bar{z}_n}{\eta(|z_n|^2 + \epsilon^2)^2} \geq \frac{\partial\rho\bar{\partial}\rho}{\eta}$ on $\text{supp } v$, we get

$$\int_{\text{supp } v} |\bar{\partial}\psi|_{\partial\bar{\partial}(\phi+\psi)}^2 |u|^2 e^{\psi-\phi} \leq \int_{\Omega} \frac{4}{\eta} |u|^2 e^{\psi-\phi}. \quad (5)$$

Substituting (3),(4),(5) into (2),

$$\int_{\Omega} \left(\frac{\frac{\eta}{(-\rho+1)^2}}{1 + \frac{\eta}{(-\rho+1)^2}} - \frac{4r}{\eta} \right) |u|^2 e^{\psi-\phi} \leq (1 + r^{-1}) C_n \int_V |f|^2 e^{-\varphi}.$$

Since $\eta \asymp -\rho$, we may choose $r = r_n$ sufficiently small such that the left side of the above inequality is bounded below by $c_n \int_{\Omega} \frac{|u|^2}{|z_n|^2 \rho^2} e^{-\varphi}$ for some constant $c_n > 0$. Now put $F_{\epsilon} = \chi(|z_n|^2/\epsilon^2)f - u$. We conclude that F_{ϵ} is a holomorphic extension of f to Ω together with the estimate

$$\int_{\Omega} \frac{|F_{\epsilon}|^2}{|z_n|^2 (-\log |z_n|^2)^2} e^{-\varphi} \leq C'_n \int_V |f|^2 e^{-\varphi}.$$

By taking a weak limit of F_{ϵ} as $\epsilon \rightarrow 0$, we get the desired extension.

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